

An interior penalty function method for solving fuzzy nonlinear programming problems

Vanaja Govindhasamy, Ganesan Kandasamy

Department of Mathematics, College of Engineering and Technology, SRM Institute of Science and Technology, Tamil Nadu, India

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ABSTRACT

In this article, we investigate fuzzy interior penalty function method for solving fuzzy nonlinear programming problems (FNLPP) based on a new fuzzy arithmetic, unconstrained optimization, and fuzzy ranking on the parametric form of triangular fuzzy numbers (TFN). The main objective of this paper is to solve constrained fuzzy nonlinear programming problems using interior penalty functions (IPF) by converting it into unconstrained optimization problems. We prove an important lemma and a convergence theorem for the interior penalty functions method. Interior penalty function techniques favor sites near the boundary of the feasible region in the interior. We present a numerical example of the suggested method and compare the results to those produced by existing methods.

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Corresponding Author:

Ganesan Kandasamy

Department of Mathematics, College of Engineering and Technology, SRM Institute of Science and Technology
Kattankulathur, Tamil Nadu 603203, India

Email: ganessank@srmist.edu.in

1. INTRODUCTION

For many years, traditional optimization techniques have been successful. Due to many reasons, real world problems involves uncertainties and inexactness. Hence to formulate and to solve real world problems, the traditional mathematical tools are inefficient. Fuzzy set was introduced by Zadeh [1]. It plays a crucial role in solving the real world problems. There after, Bellman and Zadeh [2] have discussed the concept of decision making in fuzzy nature. Fuzzy set theory and its applications was discussed by Zimmermann [3]. There are several fuzzy nonlinear production planning and scheduling issues in many real-world situations, such as in industrial planning. Due to the existence of inaccurate information, research on optimization and modeling for nonlinear programming in an environment on fuzzy is crucial for the development of fuzzy optimization theory and having a wide range of applications to a variety of practical conflicts in the real world.

Akrami and Hosseini [4] focused on solving fuzzy nonlinear optimization problems. They considered all coefficients in the constraints and objective function to be fuzzy numbers. A α -cut was used to transform the fuzzy problem into a crisp form. This crisp form becomes an interval nonlinear programming problems (NLPP), which no longer requires the use of membership functions for solving and obtained the interval solution. Behera and Nayak [5] found the fuzzy optimal solution for nonlinear programming problems with linear constraints. MA [6] developed a exact and smooth penalty function to transform nonlinear programming problems into unconstrained optimization models. The results indicate that this new penalty function is a reasonable and effective method for solving a specific type of NLPP. Antczak [7] discussed an optimization problem that involves a fuzzy objective function with inequality and equality constraints, all of which had locally Lipschitz functions. They successfully demonstrated that for this particular non-differentiable fuzzy optimization prob-

lem, they could create an associated bi-objective optimization problem. They also established a relationship between the vector optimization problem of Pareto solutions and the non-dominated (weakly) solutions of the original non-differentiable fuzzy optimization problem.

Jameel and Radhi [8] developed mixed with Nelder and Mend's algorithm of direct optimization problem and the penalty function method to solve fuzzy nonlinear programming problems (FNLPP). Gani and Yogarani [9] proposed a method to solve the primal-dual fuzzy linear programming problem more efficiently. They achieved this by combining the barrier function of fuzzy exponential with the barrier parameter of fuzzy exponential. Othman and Abdulrazzaq [10] discussed the control system for a bearingless brushless DC (BBLDC) motor used in an artificial heart pump. They conducted simulations of this system and explored recent progress in medical equipment, remote technologies, detector patterns, and ongoing developments in IoT and fuzzy logic-based healthcare. Also, an examination was conducted on existing global healthcare policies to assess their effectiveness in supporting the long-term growth of IoT and fuzzy logic in healthcare. Kanaya [11] discussed a method for solving multi-objective nonlinear programming problems that involve fuzzy parameters in the objective functions. Utilizing an interactive cutting-plane algorithm, this method is reliant on the stability set corresponding to α -Pareto optimal solutions obtained through the same method.

Kaliyaperumal and Das [12] introduced a fuzzy version of the problem, which they addressed using the necessary and sufficient conditions of Lagrangian multipliers with a focus on fuzziness. They illustrated this approach with a numerical example. By solving two numerical examples—one using membership functions (MFs) and the other using robust rankings—they clarified the model's effectiveness. This model is designed to tackle uncertainties and subjective experiences of decision-makers and can assist in resolving challenges associated with decision-making. Kemal [13] discussed interior and exterior penalty methods for finding optimal solutions of nonlinear optimization problems by reducing to unconstrained optimization problems. Kiruthiga and Hemalatha [14] studied the method for solving nonlinear programming problems with multiple objectives. They transform these problems into a goal multi-objective nonlinear programming problem, aiming to solve the corresponding min-max problem and acquire the M-Pareto optimal solution. This process also yields the membership function value, along with information on the trade-off rate between the membership functions.

Lalitha and Loganathan [15], [16] conducted research on fuzzy nonlinear programming problems. To obtain more precise solutions, they applied fuzzy sets properties and fuzzy numbers with linear membership functions, as well as a fuzzy maximum decision-making approach, to transform crisp problems into fuzzy ones. In their illustration, they provide a refinement process for obtaining these solutions. They also proposed a method to find fuzzy solutions for fully fuzzy nonlinear programming problems with inequality constraints FFNLP, which commonly occur in real-life situations. Nagoorgani and Sudha [17] discussed optimality conditions for fuzzy non-linear unconstrained minimization problems. The cost coefficients are represented by TFN and presented some numerical examples. Allibhoy and Cortés [18] introduced the 'safe gradient flow,' which is a continuous-time dynamic system used to solve constrained optimization problems. This system ensures that the feasible set remains unchanged over time. This system, derived as a continuous approximation of the projected gradient flow or by augmenting the objective function's gradient flow with additional inputs, employs a control barrier function-based quadratic program to ensure feasible set safety.

Pirzada and Pathak [19], [20] developed essential and comprehensive optimality conditions similar to Kuhn-Tucker conditions for nonlinear fuzzy optimization problems. These problems involve fuzzy-valued objective functions and fuzzy-valued constraints, and they used the concepts of convexity and H-differentiability for fuzzy-valued functions. Additionally, they recommended using the Newton's method to discover non-dominated solutions for unconstrained multi-variable fuzzy optimization problems. Shankar *et al.* [21] used a genetic algorithm (GA) to solve fuzzy nonlinear optimization problems. They tackled this fuzzy problem by defining membership for fuzzy numbers. In their approach, they represented each fuzzy number by dividing it into specific partition points. The resulting values, obtained through an evolutionary process, represent the membership level of the fuzzy number. What's notable is that using GA to compute fuzzy equations eliminates the need for the traditional extension principle, interval arithmetic, or α -cuts typically required for solving fuzzy nonlinear programming problems.

Salman and Jilawi [22] introduced penalty function methods for solving optimization problems with constraints. The methods they discussed aim to transform a optimization problem from constrained to an unconstrained one. After this transformation, they apply standard search techniques like the exterior penalty function method and the interior penalty method to find solutions. Saranya and Kaliyaperumal [23] proposed a new approach for solving NLPP in terms of fuzziness using trapezoidal fuzzy membership functions (TFMFs)

and its arithmetic operations and also stated Kuhn Tuckers necessary and sufficient condition in terms of fuzziness for finding alpha optimal solution of the problem. Maheswari and Ganesan [24], [25] proposed a fuzzy version of the Kuhn-Tucker condition for fully fuzzy nonlinear programming problems and found their optimal fuzzy solutions. They used the Gradient method (also known as the Steepest Descent method of Cauchy) to convert it into an unconstrained multi-variable fuzzy optimization problem. They also transformed the FNLPP with TFN into its parametric form and applied the fuzzy version of the Karush-Kuhn-Tucker necessary and sufficient conditions for optimality, which helped them obtain non-dominated solutions for a NLPP involving TFN.

Lu and Liu [26] introduced the concept of signal-to-noise (S/N) ratio and it involves formulating a pair of nonlinear fractional programs to determine the lower and upper bounds of the fuzzy S/N ratio with fuzzy observations. Model reduction and variable substitutions are used to transform these programs into a pair of quadratic programs, which can then be solved. This approach allows for an accurate assessment of the manufacturing processes. A methodology is developed to evaluate the manufacturing processes for a company in Taiwan. Puri and Ralescu [27] extended the definition of differential to fuzzy functions, it is natural to extend first the Ridstrom embedding theorem by considering an appropriate space of fuzzy subsets of a Banach space. Singh and Yadav [28] proposed a novel approach for NLPPs in intuitionistic fuzzy environments, accommodating both partial and full parameter fuzziness through a component-wise optimization and non-linear membership function-based transformation. Subsequently, this fuzzy model is employed to convert into a crisp model.

The main contribution of this research paper is as follows: Most of the author's have transformed the FNLPP into one or more equivalent crisp nonlinear programming problems and obtained the crisp solution. By using a new fuzzy arithmetic and ranking on the parametric form of the triangular fuzzy numbers and by using the interior penalty functions method, we obtained the fuzzy optimal solution of the given FNLPPs without converting to its equivalent crisp form. We prove Barrier's Lemma and the Convergence theorem for the interior penalty functions method. A numerical example provided to show the efficacy of the proposed method and the results are compared with the existing ones. The results of the methods are shown graphically.

2. RESEARCH METHOD

In this section, we recall some essential primary concepts and backgrounds on fuzzy numbers which are most required.

Definition 1 A fuzzy number \tilde{S} is a fuzzy set on R whose membership function $\tilde{S} : R \rightarrow [0, 1]$ has the following characteristics:

- \tilde{S} is convex, i.e., $\tilde{S}(\lambda y_1 + (1 - \lambda)y_2) \geq \min\{\tilde{S}(y_1), \tilde{S}(y_2)\}$, $\lambda \in [0, 1]$, for all $y_1, y_2 \in R$.
- \tilde{S} is normal, i.e., there exists an $y \in R$ such that $\tilde{S}(y) = 1$
- \tilde{S} is piecewise continuous.

We use the notation $F(R)$ to denote the set of all fuzzy numbers defined on R .

Definition 2 A triangular fuzzy number (TFN) \tilde{S} is a fuzzy number \tilde{S} on R , where $\tilde{S} : R \rightarrow [0, 1]$ satisfies:

$$\tilde{S}(y) = \begin{cases} \frac{y - s_1}{s_2 - s_1}, & s_1 \leq y \leq s_2 \\ \frac{s_3 - y}{s_3 - s_2}, & s_2 \leq y \leq s_3 \\ 0, & \text{elsewhere} \end{cases}$$

We denote this triangular fuzzy number by $\tilde{S} = (s_1, s_2, s_3)$.

Definition 3 A fuzzy number \tilde{S} can also be represented as a pair $(\underline{s}; \overline{s})$ of functions $\underline{s}(\alpha)$, $\overline{s}(\alpha)$, $0 \leq \alpha \leq 1$ which satisfy the following requirements:

- $\underline{s}(\alpha)$ is a bounded monotonic increasing left continuous function.
- $\overline{s}(\alpha)$ is a bounded monotonic decreasing left continuous function.
- $\underline{s}(\alpha) \leq \overline{s}(\alpha)$, $0 \leq \alpha \leq 1$.

Definition 4 Let $\tilde{S} = (s_1, s_2, s_3)$ be a triangular fuzzy number and $\underline{s}(\alpha) = s_1 + (s_2 - s_1)\alpha$, $\overline{s}(\alpha) = s_3 - (s_3 - s_2)\alpha$, $\alpha \in [0, 1]$. The parametric form of the TFN is defined as $\tilde{S} = (s_0, s_*, s^*)$, where $s_* = s_0 - \underline{s}$

and $s^* = \bar{s} - s_0$ are the left and right fuzziness index functions respectively. The number $s_0 = \left(\frac{\underline{s}(1) + \bar{s}(1)}{2} \right)$ is called the location index number. When $\alpha = 1$, we get $s_0 = s_2$.

2.1. Arithmetic operations on fuzzy numbers

Ma *et al.* [29] have expressed all the fuzzy numbers in their parametric form, i.e. in the form of location index and fuzziness index functions. They suggested arithmetic operations on fuzzy numbers in which the location index numbers obey the classical arithmetic and the fuzziness index functions obey the Lattice rule. That is for s, q in Lattice L , $s \vee q = \max\{s, q\}$ and $s \wedge q = \min\{s, q\}$. For any two fuzzy numbers $\tilde{S} = (s_0, s_*, s^*)$, $\tilde{Q} = (q_0, q_*, q^*)$ and $*$ $\in \{+, -, \times, \div\}$, the arithmetic operations are defined as

$$\begin{aligned}\tilde{S} * \tilde{Q} &= (s_0, s_*, s^*) * (q_0, q_*, q^*) = (s_0 * q_0, s_* \vee q_*, s^* \vee q^*) \\ &= (s_0 * q_0, \max\{s_*, q_*\}, \max\{s^*, q^*\})\end{aligned}$$

In particular for $\tilde{S} = (s_0, s_*, s^*)$ and $\tilde{Q} = (q_0, q_*, q^*)$ in $F(R)$, we have:

- Addition: $\tilde{S} + \tilde{Q} = (s_0, s_*, s^*) + (q_0, q_*, q^*) = (s_0 + q_0, \max\{s_*, q_*\}, \max\{s^*, q^*\})$
- Subtraction: $\tilde{S} - \tilde{Q} = (s_0, s_*, s^*) - (q_0, q_*, q^*) = (s_0 - q_0, \max\{s_*, q_*\}, \max\{s^*, q^*\})$
- Multiplication: $\tilde{S} \times \tilde{Q} = (s_0, s_*, s^*) \times (q_0, q_*, q^*) = (s_0 \times q_0, \max\{s_*, q_*\}, \max\{s^*, q^*\})$
- Division: $\tilde{S} \div \tilde{Q} = (s_0, s_*, s^*) \div (q_0, q_*, q^*) = (s_0 \div q_0, \max\{s_*, q_*\}, \max\{s^*, q^*\})$, provided $q_0 \neq 0$.

2.2. Ranking of fuzzy numbers

Ranking of fuzzy numbers play major role in decision making process under fuzzy environment. Different types of ranking methods suggested by several authors are available in the literature. In this article, we use an efficient ranking method based on the graded mean.

$$\text{For } \tilde{S} = (s_0, s_*, s^*) \in F(R), \text{ define } \Re : F(R) \rightarrow R \text{ by } \Re(\tilde{S}) = \left(\frac{s_* + 4s_0 + s^*}{6} \right).$$

For any two triangular fuzzy numbers $\tilde{S} = (s_0, s_*, s^*)$ and $\tilde{Q} = (q_0, q_*, q^*)$ in $F(R)$, we have the following comparison:

- If $\Re(\tilde{S}) < \Re(\tilde{Q})$, then $\tilde{S} \prec \tilde{Q}$
- If $\Re(\tilde{S}) > \Re(\tilde{Q})$, then $\tilde{S} \succ \tilde{Q}$
- If $\Re(\tilde{S}) = \Re(\tilde{Q})$, then $\tilde{S} \approx \tilde{Q}$.

Definition 5 [30] A fuzzy-valued function $\tilde{f} : R \rightarrow F(R)$ is said to be continuous at $t_0 \in R$, if for a given $\epsilon > 0$, there exist $\delta > 0$ such that $D(\tilde{f}(t), \tilde{f}(t_0)) < \epsilon$ whenever $|t - t_0| < \delta$.

Definition 6 [31] for arbitrary fuzzy numbers $\tilde{S} = (\underline{s}, \bar{s})$ and $\tilde{Q} = (\underline{q}, \bar{q})$, the quantity:

$$D(\tilde{S}, \tilde{Q}) = \sup_{0 \leq r \leq 1} \{ \max[|\underline{s}(\alpha) - \underline{q}(\alpha)|, |\bar{s}(r) - \bar{q}(r)|] \} \quad (1)$$

is called the distance(D) between \tilde{S} and \tilde{Q} .

Definition 7 [31] Let $\tilde{f} : R \rightarrow F(R)$ be a fuzzy-valued function and let $t_0 \in R$. The derivative $\tilde{f}'(t_0)$ of \tilde{f} at the point t_0 is defined by

$$\tilde{f}'(t_0) = \lim_{h \rightarrow 0} \frac{\tilde{f}(t_0 + h) - \tilde{f}(t_0)}{h} \quad (2)$$

provided that this limit taken with respect to the metric D , exists.

Theorem 1 [31] Let $\tilde{f} : [s, q] \rightarrow F(R)$ and its derivative exist. Then:

- the derivative of $\tilde{f}_0(t)$ exists and $[\tilde{f}'(t)]_0 = \tilde{f}'_0(t)$;
- $(\tilde{f}'(t))^* \leq \inf_{h>0} \sup_{\alpha \in [-h, h]} \{(\tilde{f}(t + \alpha))^*\}$;
- $(\tilde{f}'(t))^* \leq \inf_{h>0} \sup_{\alpha \in [-h, h]} \{(\tilde{f}(t + \alpha))^*\}$.

2.3. Fuzzy non linear programming problems

Consider a general fuzzy nonlinear programming problem:

$$\begin{aligned} & \min \tilde{f}(y) \\ & \text{subject to } \tilde{h}_i(y) \approx \tilde{0} \text{ for } i = 1, 2, \dots, l \\ & \quad \tilde{g}_j(y) \preceq \tilde{0} \text{ for } j = 1, 2, \dots, m, \\ & \quad \mathbf{y} \succeq \tilde{0} \end{aligned} \quad (3)$$

where $\tilde{f}, \tilde{h}_1, \dots, \tilde{h}_l, \tilde{g}_1, \dots, \tilde{g}_m$ are continuous fuzzy valued functions defined on R^n .

A vector $\mathbf{y} = (y_1, y_2, y_3, \dots, y_n)$ is said to be a feasible solution to the FNLPP if it satisfies the constraints of the FNLPP. The set of all feasible solutions forms the feasible region and is defined by

$$F = \{\mathbf{y} \in R^n / \tilde{h}_i(y) \approx \tilde{0}, \text{ for } i = 1, 2, 3, \dots, l, \tilde{g}_j(y) \preceq \tilde{0}, \text{ for } j = 1, 2, 3, \dots, m\}.$$

Even for problems with linear constraints, nonlinear programmes (NLPP) generally differ significantly from linear programmes. For a NLPP, the optimal solution can be attained

- at a point inside the feasible region,
- at a location on the feasible region's boundary that is not an extreme point, or
- at a point on the feasible region's extreme boundary.

As a result, methods that only look for extreme points, such the simplex method, may not find the best solution for NLPP. We therefore search for another effective FNLPP approaches.

2.4. Fuzzy interior penalty function method

We present a fuzzy version of the interior penalty function approach for solving FNLPP. We prove Convergence of fuzzy valued interior penalty function method (Barrier's Lemma) and the convergence theorem for the solution of FNLPP.

Definition 8 A continuous fuzzy valued function $\tilde{P} : R^n \rightarrow F(R)$ is said to be a fuzzy valued penalty function if \tilde{P} satisfies:

- $\tilde{P}(y) \approx \tilde{0}$ if and only if $\tilde{g}_i(y) \preceq \tilde{0}$
- $\tilde{P}(y) \succ \tilde{0}$ for all $\tilde{g}_i(y) \not\preceq \tilde{0}$

The fuzzy valued penalty approach is based on a new parameter called "penalty term". In this approach, a new fuzzy optimization problem is formed by introducing a parameter (penalty term) in objective function in such a way that any constraint violation will leads to imposing of large penalty. This new parameter gives the values that are very huge and convenient. This new fuzzy optimization problem is similar to the given fuzzy optimization problem. If there are only inequality constraints in fuzzy optimization problems, the following type of fuzzy valued penalty function is frequently used: $\tilde{P}(y) = \sum_{i=1}^m (\max\{0, \tilde{g}_i(y)\})^p$, where p is a positive integer.

Definition 9 consider a FNLPP with inequality constraints:

$$\begin{aligned} & \min \tilde{f}(y) \\ & \text{subject to } \tilde{g}_i(y) \preceq \tilde{0} \quad \text{for } i = 1, 2, 3, \dots, m \\ & \quad \mathbf{y} \succeq \tilde{0}, \end{aligned} \quad (4)$$

where $\tilde{f}, \tilde{g}_1, \dots, \tilde{g}_m$ are continuous fuzzy valued functions defined on R^n .

Let ∞ as the boundary is approached from the interior and the non-negative over the interior of the set $\{\mathbf{y} | \tilde{g}(y) \prec \tilde{0}\}$ and fuzzy valued interior penalty function \tilde{P} is continuous.

Let $\omega(z) \geq 0$ if $z < 0$ and $\lim_{z \rightarrow 0^-} \omega(z) \rightarrow \infty$, where ω is a continuous univariate function over $\{z : z < 0\}$ and $z = \tilde{g}_i(y)$. Then

$$\tilde{P}(y) = \sum_{i=1}^m \omega[\tilde{g}_i(y)].$$

Fuzzy valued interior penalty functions are classified into two types:

- Inverse fuzzy valued function:

$$\tilde{P}(y) = -\sum_{i=1}^m \frac{1}{\tilde{g}_i(y)}; \text{ for } \{\tilde{g}_i(y) \prec \tilde{0}\}$$

- Logarithmic fuzzy valued function:

$$\tilde{P}(y) = -\sum_{i=1}^m \log[-\tilde{g}_i(y)]; \text{ for } \{\tilde{g}_i(y) \prec \tilde{0}\}$$

It is to be noted that $\lim_{\tilde{g}_i(y) \rightarrow 0} \tilde{P}(y) \rightarrow \infty$ in both the cases.

We define an auxiliary function $\tilde{\chi}_{\mu\kappa}(y) = \tilde{f}(y) + \mu\tilde{P}(y)$, where μ is a positive constant. Now we want $\tilde{P}(y) \approx \tilde{0}$ if $\tilde{g}_i(y) \preceq \tilde{0}$ and $\tilde{P}(y) \rightarrow \infty$ if $\tilde{g}_i(y) \rightarrow 0$ so that we retain the region $\{y | \tilde{g}(y) \preceq \tilde{0}\}$. The unconstrained optimization has considerable computational challenges as $\tilde{P}(y)$ is discontinuous now. This ideal form of \tilde{P} is replaced by a more realistic requirement that \tilde{P} be non negative continuous over the region $\{y | \tilde{g}(y) \prec \tilde{0}\}$ and approaches infinity as it access the limit from within. The barrier (penalty) function need not be defined at impossible locations. The barrier problem corresponding to the FNLPP (4) is expressed as (5):

$$\begin{aligned} & \min \tilde{f}(y) + \mu_{\kappa}\tilde{P}(y) \\ & \text{subject to } \tilde{g}_i(y) \preceq \tilde{0} \text{ for } i = 1, 2, 3, \dots, m \\ & \quad y \succeq \tilde{0}. \end{aligned} \quad (5)$$

It is a constrained fuzzy optimization problem. The advantage of this barrier is that it may be overcome using an unconstrained search strategy that begins at a starting interior point and then searches using steepest descent or further iterative descent method suited for unconstrained problems. From a computational viewpoint, the barrier problem is an unconstrained fuzzy optimization problem, considering the fact that it appears to be a constrained fuzzy optimization problem.

2.5. Convergence of fuzzy valued interior penalty function method

Starting with μ_1 we produce a series of points. The sequence μ_{κ} satisfy $\mu_{\kappa+1} < \mu_{\kappa}$ and $\mu_{\kappa} \rightarrow 0$ as $\kappa \rightarrow \infty$. Let y^{κ} be any feasible solution of the problem (5) and y^* be the optimal solution (5). The next lemma provides some fundamental characteristics of barrier models.

Lemma 1 (Barrier Lemma).

- $\tilde{\chi}_{\mu\kappa}(y^{\kappa}) \succeq \tilde{\chi}_{\mu\kappa+1}(y^{\kappa+1})$
- $\tilde{P}(y^{\kappa}) \preceq \tilde{P}(y^{\kappa+1})$
- $\tilde{f}(y^{\kappa}) \succeq \tilde{f}(y^{\kappa+1})$
- $\tilde{f}(y^*) \preceq \tilde{f}(y^{\kappa}) \preceq \tilde{\chi}_{\mu\kappa}(x^{\kappa})$

Proof:

- Let $\tilde{\chi}_{\mu\kappa}(y^{\kappa}) = \tilde{f}(y^{\kappa}) + \mu_{\kappa}\tilde{P}(y^{\kappa}) \succeq \tilde{f}(y^{\kappa}) + \mu_{\kappa+1}\tilde{P}(y^{\kappa}) \succeq \tilde{f}(y^{\kappa+1}) + \mu_{\kappa+1}\tilde{P}(y^{\kappa+1}) \approx \tilde{\chi}_{\mu\kappa+1}(y^{\kappa+1}) \Rightarrow \tilde{\chi}_{\mu\kappa}(y^{\kappa}) \succeq \tilde{\chi}_{\mu\kappa+1}(y^{\kappa+1})$.
- Let $\tilde{f}(y^{\kappa}) + \mu_{\kappa}\tilde{P}(y^{\kappa}) \preceq \tilde{f}(y^{\kappa+1}) + \mu_{\kappa}\tilde{P}(y^{\kappa+1})$ and $\tilde{f}(y^{\kappa+1}) + \mu_{\kappa+1}\tilde{P}(y^{\kappa+1}) \preceq \tilde{f}(y^{\kappa}) + \mu_{\kappa+1}\tilde{P}(y^{\kappa})$. We have, $\tilde{f}(y^{\kappa}) + \mu_{\kappa}\tilde{P}(y^{\kappa}) - \tilde{f}(y^{\kappa+1}) - \mu_{\kappa}\tilde{P}(y^{\kappa+1}) - \tilde{f}(y^{\kappa+1}) + \mu_{\kappa+1}\tilde{P}(y^{\kappa+1}) + \mu_{\kappa+1}\tilde{P}(y^{\kappa}) \preceq \tilde{0} \Rightarrow (\mu_{\kappa} - \mu_{\kappa+1})\tilde{P}(y^{\kappa}) - (\mu_{\kappa} - \mu_{\kappa+1})\tilde{P}(y^{\kappa+1}) \preceq \tilde{0} \Rightarrow \tilde{P}(y^{\kappa}) \preceq \tilde{P}(y^{\kappa+1})$.
- Using *i* and *ii*, we simply get $\tilde{f}(y^{\kappa}) \succeq \tilde{f}(y^{\kappa+1})$.
- When y^* is an optimal solution, we have $\tilde{f}(y^*) \preceq \tilde{f}(y^{\kappa})$ for any feasible solution y^{κ} . Therefore $\tilde{f}(y^*) \preceq \tilde{f}(y^{\kappa}) \preceq \tilde{f}(y^{\kappa}) + \mu_{\kappa}\tilde{P}(y^{\kappa}) \approx \tilde{\chi}_{\mu\kappa}(y^{\kappa}) \Rightarrow \tilde{f}(y^*) \preceq \tilde{\chi}_{\mu\kappa}(y^{\kappa})$.

Theorem 2 (Convergence Theorem) Let $\tilde{g}(y)$, $\tilde{P}(y)$ and $\tilde{f}(y)$ be continuous fuzzy valued functions. Let y^{κ} , for $\kappa = 1, 2, 3, \dots$, be a series of solutions of $\tilde{\chi}_{\mu\kappa}(y)$. If there exists an optimum solution y^* of (5) therefore $\{\tau \cap y | \tilde{g}(y) \prec \tilde{0}\} \neq \emptyset$, where τ is a neighborhood of y^* . Then any limit point y of y^{κ} solves (5). Moreover,

$\mu_\kappa \tilde{P}(y) \rightarrow 0$ as $\mu_\kappa \rightarrow 0$.

Proof: Let the sequence $\{y^\kappa\}$ be any limit point for y . From the continuity of $\tilde{f}(y)$ and $\tilde{g}(y)$, $\lim_{\kappa \rightarrow \infty} \tilde{f}(y^\kappa) = \tilde{f}(y)$ and $\lim_{\kappa \rightarrow \infty} \tilde{g}(y^\kappa) \approx \tilde{g}(y) \preceq \tilde{0}$. Thus y is a feasible point for (5).

Given any $\epsilon > 0$, there exists y such that $\tilde{g}(y) \prec \tilde{0}$ and $\tilde{f}(y) \preceq \tilde{f}(y^*) + \epsilon$. For each κ ,

$$\tilde{f}(y^*) + \epsilon + \mu_\kappa \tilde{P}(y) \succeq \tilde{f}(y) + \mu_\kappa \tilde{P}(y) \succeq \tilde{\chi}_{\mu_\kappa}(y^\kappa).$$

Therefore, for sufficiently large κ , $\tilde{f}(y^*) + 2\epsilon \succeq \tilde{\chi}_{\mu_\kappa}(y^\kappa)$ and since $\tilde{\chi}_{\mu_\kappa}(y^\kappa) \succeq \tilde{f}(y^*)$ from *iv* of lemma (1), we have $\tilde{f}(y^*) + 2\epsilon \succeq \lim_{\kappa \rightarrow \infty} \tilde{\chi}_{\mu_\kappa}(y^\kappa) \succeq \tilde{f}(y^*)$. This implies that, $\lim_{\kappa \rightarrow \infty} \tilde{\chi}_{\mu_\kappa}(y^\kappa) = \tilde{f}(y) + \lim_{\kappa \rightarrow \infty} \mu_\kappa \tilde{P}(y^\kappa) = \tilde{f}(y^*)$. We also have, $\tilde{f}(y^*) \preceq \tilde{f}(y^\kappa) \preceq \tilde{f}(y^\kappa) + \mu_\kappa \tilde{P}(y^\kappa) = \tilde{\chi}_{\mu_\kappa}(y^\kappa)$. Taking limits we obtain $\tilde{f}(y^*) \preceq \tilde{f}(y) \preceq \tilde{f}(y^*)$. From this, we have $\tilde{f}(y^*) = \tilde{f}(y)$. Hence, $\tilde{f}(y)$ is the optimal solution of the original nonlinear inequality constrained problem (5). Furthermore, from $\tilde{f}(y) + \lim_{\kappa \rightarrow \infty} \mu_\kappa \tilde{P}(y^\kappa) = \tilde{f}(y^*)$, we have

$$\lim_{\kappa \rightarrow \infty} \mu_\kappa \tilde{P}(y^\kappa) = \tilde{f}(y^*) - \tilde{f}(y) = 0.$$

\therefore As $\kappa \rightarrow \infty$, i.e., $\mu_\kappa \rightarrow 0$, the function $\mu_\kappa \tilde{P}(y^\kappa) \rightarrow 0$ for each κ .

2.6. Algorithm for fuzzy valued interior penalty function method

The fuzzy valued interior penalty function method is a powerful numerical technique used in optimization and constraint handling problems, particularly in the field of engineering and mathematical modeling. This algorithmic approach provides a systematic and efficient means of tackling optimization challenges where traditional methods may fall short due to uncertainty or imprecision in input parameters or constraints.

Step 1. Assume a growth parameter $\gamma > 1$, a stopping parameter (tolerance) $\epsilon > 0$ and an initial value $\mu_1 > 0$. Let y^κ be an initial feasible solution with $\tilde{g}(y)^\kappa \prec \tilde{0}$ and formulate the objective function $\tilde{\chi}_{\mu_\kappa}(y)$, where $\kappa = 1, 2, 3, \dots, n$.

Step 2. Starting with y^κ , apply an unconstrained search technique to find the point that minimizes $\tilde{\chi}_{\mu_\kappa}(y)$ and call it $y^{\kappa+1}$, the new starting point.

Step 3. If $\|y^{\kappa+1} - y^\kappa\| < \epsilon$, then stop with $y^{\kappa+1}$ an estimate of the optimal solution. Otherwise, put $\mu_{\kappa+1} = \gamma\mu_\kappa$ and formulate the new $\tilde{\chi}_{\mu_{\kappa+1}}(y)$ and put $\kappa = \kappa + 1$ and return to step 1.

The interior penalty function method prefers locations near the border and inside the feasible region.

2.7. Numerical example

Consider a FNLPP discussed by Kemal [13]:

$$\begin{aligned} \min \tilde{f}(\tilde{y}) &= (0, 1, 2)\tilde{y}_1^2 + (1, 2, 3)\tilde{y}_2^2 \\ \text{subject to } \tilde{g}(\tilde{y}) &= (0, 1, 2) - (0, 1, 2)\tilde{y}_1 - (0, 1, 2)\tilde{y}_2 \preceq \tilde{0} \\ \tilde{y} &\succeq \tilde{0}. \end{aligned} \quad (6)$$

Solution: The parametric form of the given FNLPP (6) is given by:

$$\begin{aligned} \min \tilde{f}(\tilde{y}) &= (1, 1 - \alpha, 1 - \alpha)\tilde{y}_1^2 + (2, 1 - \alpha, 1 - \alpha)\tilde{y}_2^2 \\ \text{subject to } \tilde{g}(\tilde{y}) &= (1, 1 - \alpha, 1 - \alpha) - (1, 1 - \alpha, 1 - \alpha)\tilde{y}_1 - (1, 1 - \alpha, 1 - \alpha)\tilde{y}_2 \preceq \tilde{0} \\ \tilde{y} &\succeq \tilde{0}, \quad \alpha \in [0, 1] \end{aligned}$$

Define the barrier function as:

$$\tilde{P}(\tilde{y}) = -\log[-\tilde{g}(\tilde{y})] \Rightarrow -\log[(1, 1 - \alpha, 1 - \alpha)\tilde{y}_1 + (1, 1 - \alpha, 1 - \alpha)\tilde{y}_2 - (1, 1 - \alpha, 1 - \alpha)]$$

Then the corresponding unconstrained fuzzy optimization problem is:

$$\begin{aligned} \min \tilde{\chi}_{\mu_\kappa} &= \tilde{f}(\tilde{y}) + \mu_\kappa \tilde{P}(\tilde{y}) \\ &= (1, 1 - \alpha, 1 - \alpha)\tilde{y}_1^2 + (2, 1 - \alpha, 1 - \alpha)\tilde{y}_2^2 \\ &\quad - \mu_\kappa \log[(1, 1 - \alpha, 1 - \alpha)\tilde{y}_1 + (1, 1 - \alpha, 1 - \alpha)\tilde{y}_2 - (1, 1 - \alpha, 1 - \alpha)] \end{aligned}$$

The necessary condition for \tilde{y} to be optimal implies $\nabla \tilde{f}(\tilde{y}) \approx \tilde{0}$. Hence we have:

$$\frac{\partial \tilde{\chi}_{\mu\kappa}}{\partial \tilde{y}_1} = (2, 1 - \alpha, 1 - \alpha) \tilde{y}_1 - \frac{\mu_{\kappa}(1, 1 - \alpha, 1 - \alpha)}{[(1, 1 - \alpha, 1 - \alpha) \tilde{y}_1 + (1, 1 - \alpha, 1 - \alpha) \tilde{y}_2 - (1, 1 - \alpha, 1 - \alpha)]} \approx \tilde{0}.$$

$$\frac{\partial \tilde{\chi}_{\mu\kappa}}{\partial \tilde{y}_2} = (4, 1 - \alpha, 1 - \alpha) \tilde{y}_2 - \frac{\mu_{\kappa}(1, 1 - \alpha, 1 - \alpha)}{[(1, 1 - \alpha, 1 - \alpha) \tilde{y}_1 + (1, 1 - \alpha, 1 - \alpha) \tilde{y}_2 - (1, 1 - \alpha, 1 - \alpha)]} \approx \tilde{0}.$$

Solving these equations, we get,

$$\tilde{y}_1 \approx \frac{(1, 1 - \alpha, 1 - \alpha) \pm \sqrt{(1, 1 - \alpha, 1 - \alpha) + (3\mu_{\kappa}, 1 - \alpha, 1 - \alpha)}}{(3, 1 - \alpha, 1 - \alpha)}$$

$$\text{and } \tilde{y}_2 \approx \frac{(1, 1 - \alpha, 1 - \alpha) \pm \sqrt{(1, 1 - \alpha, 1 - \alpha) + (3\mu_{\kappa}, 1 - \alpha, 1 - \alpha)}}{(6, 1 - \alpha, 1 - \alpha)}$$

Negative signs make it infeasible, thus we have:

$$\tilde{y}_1 \approx \frac{(1, 1 - \alpha, 1 - \alpha) + \sqrt{(1, 1 - \alpha, 1 - \alpha) + (3\mu_{\kappa}, 1 - \alpha, 1 - \alpha)}}{(3, 1 - \alpha, 1 - \alpha)}$$

$$= \left(\frac{(1 + \sqrt{3\mu_{\kappa} + 1})}{3}, 1 - \alpha, 1 - \alpha \right)$$

$$\text{and } \tilde{y}_2 \approx \frac{(1, 1 - \alpha, 1 - \alpha) + \sqrt{(1, 1 - \alpha, 1 - \alpha) + (3\mu_{\kappa}, 1 - \alpha, 1 - \alpha)}}{(6, 1 - \alpha, 1 - \alpha)}$$

$$= \left(\frac{(1 + \sqrt{3\mu_{\kappa} + 1})}{6}, 1 - \alpha, 1 - \alpha \right)$$

Let $\mu_{\kappa+1} = \gamma\mu_{\kappa}$. Starting with $\gamma = 0.1$, $\mu_1 = 1$ and $\tilde{\mathbf{y}}^1 = [(1, 1 - \alpha, 1 - \alpha), (0.5, 1 - \alpha, 1 - \alpha)]$ and using a tolerance of 0.005 (say), we have Tables 1 and 2.

Table 1. Barrier iteration table

κ	μ_{κ}	$\tilde{\mathbf{y}}_1^{\kappa}$	$\tilde{\mathbf{y}}_2^{\kappa}$	$\tilde{g}(\tilde{\mathbf{y}}^{\kappa})$	$\tilde{P}(\tilde{\mathbf{y}}^{\kappa})$
1	1	(1, 1- α , 1- α)	(0.5, 1- α , 1- α)	-(0.5, 1- α , 1- α)	(0.30103, 1- α , 1- α)
2	0.1000	(0.714, 1- α , 1- α)	(0.357, 1- α , 1- α)	-(0.0701, 1- α , 1- α)	(1.1543, 1- α , 1- α)
3	0.0100	(0.672, 1- α , 1- α)	(0.336, 1- α , 1- α)	-(0.008, 1- α , 1- α)	(2.0969, 1- α , 1- α)
4	0.0010	(0.6672, 1- α , 1- α)	(0.3336, 1- α , 1- α)	-(0.0008, 1- α , 1- α)	(3.0969, 1- α , 1- α)
5	0.0001	(0.6667, 1- α , 1- α)	(0.33336, 1- α , 1- α)	-(0.00006, 1- α , 1- α)	(4.2218, 1- α , 1- α)
6	0.00001	(0.666671, 1- α , 1- α)	(0.333335, 1- α , 1- α)	-(0.000006, 1- α , 1- α)	(5.2218, 1- α , 1- α)

Table 2. Continuation to Table 1

	$\mu_{\kappa} \tilde{P}(\tilde{\mathbf{y}}^{\kappa})$	$\tilde{f}(\tilde{\mathbf{y}}^{\kappa})$	$\chi_{\mu\kappa}(\tilde{\mathbf{y}}^{\kappa})$
1	(0.30103, 1- α , 1- α)	(1.5, 1- α , 1- α)	(1.80103, 1- α , 1- α)
2	(0.11543, 1- α , 1- α)	(0.7634, 1- α , 1- α)	(0.87883, 1- α , 1- α)
3	(0.02097, 1- α , 1- α)	(0.6774, 1- α , 1- α)	(0.69837, 1- α , 1- α)
4	(0.003097, 1- α , 1- α)	(0.668, 1- α , 1- α)	(0.671097, 1- α , 1- α)
5	(0.00042, 1- α , 1- α)	(0.6668, 1- α , 1- α)	(0.66722, 1- α , 1- α)
6	(0.000052, 1- α , 1- α)	(0.666676, 1- α , 1- α)	(0.6667, 1- α , 1- α)

From Tables 1 and 2, at each iteration, we observe that all the feasible points are located within the interior of the feasible region, and the final solution itself remains within this interior region. Since the barrier method converges at the 6th iteration, the optimal solution is, $\tilde{y}_1 = (0.6667, 1 - \alpha, 1 - \alpha)$, $\tilde{y}_2 = (0.3333, 1 - \alpha, 1 - \alpha)$ with $\tilde{f}(\tilde{\mathbf{y}}) = (0.66667, 1 - \alpha, 1 - \alpha)$.

That is the optimal solution of the fuzzy nonlinear programming problem (6) is $\tilde{\mathbf{y}}_1 = (a_1, a_2, a_3) = (-0.3333 + \alpha, 0.6667, 1.6667 - \alpha)$, $\tilde{\mathbf{y}}_2 = (b_1, b_2, b_3) = (-0.6667 + \alpha, 0.3333, 1.3333 - \alpha)$ with $\tilde{f}(\tilde{\mathbf{y}}) = (-0.3333 + \alpha, 0.6667, 1.6667 - \alpha)$.

3. RESULT AND DISCUSSION

Table 3 demonstrates how quickly the best solution can be reached using the proposed algorithm for fuzzy valued interior penalty function method. Table 3 and Figure 1 indicates the fuzzy optimal solution of the FNLPP (6) for different values of α .

Table 3. Fuzzy optimal solution for different values of $\alpha \in [0, 1]$

α ,	\tilde{y}_1	\tilde{y}_2	$\tilde{f}(\tilde{y})$
0	(-0.3333, 0.6667, 1.6667)	(-0.6667, 0.3333, 1.3333)	(-0.3333, 0.6667, 1.6667)
0.25	(-0.0833, 0.6667, 1.4167)	(-0.4167, 0.3333, 1.0833)	(-0.0833, 0.6667, 1.4167)
0.5	(0.1667, 0.6667, 1.1667)	(-0.1667, 0.3333, 0.8333)	(0.1667, 0.6667, 1.1667)
0.75	(0.4167, 0.6667, 0.9167)	(0.0833, 0.3333, 0.5833)	(0.4167, 0.6667, 0.9167)
1	(0.6667, 0.6667, 0.6667)	(0.3333, 0.3333, 0.3333)	(0.6667, 0.6667, 0.6667)

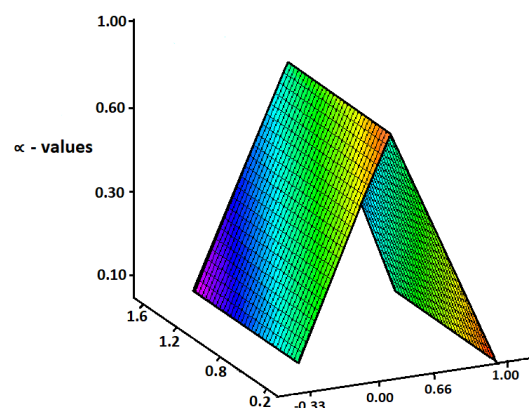


Figure 1. Fuzzy optimal solution of the FNLPP

When $\alpha = 1$, we see that $\tilde{y}_1 = 0.6667$, $\tilde{y}_2 = 0.3333$, $\tilde{f}(\tilde{y}) = 0.6667$. This solution is same as the crisp optimal solution $y_1 = 2/3$, $y_2 = 1/3$ with $\min \tilde{f}(\tilde{y}) = 2/3$ obtained by Kemal [13]

4. CONCLUSION




In this paper, we have discussed a solution concept for FNLPP involving triangular fuzzy numbers. First, the given FNLPP is expressed in terms of its location index number, left and right fuzziness index functions. On the parametric forms of fuzzy numbers, a new type of fuzzy arithmetic and fuzzy ranking are used. Barrier's Lemma and the Convergence theorem for FNLPP are established. The fuzzy version of the interior penalty function method is used, and the fuzzy optimal solution of the FNLPP is obtained without having to translate the given problem to its corresponding crisp problem. A numerical example discussed by Kemal Kiflu is solved and the solution is compared. The fuzzy optimal solution of the given FNLPP is tabulated for different values of $\alpha \in [0, 1]$ and its graphical representation is presented. It is important to note that by utilizing the suggested procedure and selecting an appropriate $\alpha \in [0, 1]$, the decision maker has the flexibility to select his/her preferred optimal solution based on the situation. The numerical solutions of crisp problems have been compared with the fuzzy solution and their effectiveness has been presented and discussed.

REFERENCES




- [1] L. A. Zadeh, "Fuzzy sets," *Information and Control*, vol. 8, no. 3, pp. 338-353, 1965, doi: 10.1016/S0019 - 9958(65)90241-X.
- [2] R. E. Bellman and L. A. Zadeh, "Decision making in a fuzzy environment," *Management Science*, vol. 17, no. 4, pp. 141-164, 1970, doi: 10.1287/mnsc.17.4.B141.
- [3] H. J. Zimmermann, "Fuzzy set theory and its applications," 4th ed. Dordrecht: Springer Netherlands, 2001, doi: 10.1007/978-94-010-0646-0.

- [4] A. Akrami and M. M. Hosseini, "An efficient method for solving a class of nonlinear fuzzy optimization problems," *Journal of Mathematical Extension*, vol. 10, no. 2, pp. 101-116, Apr. 2016.
- [5] S. K. Behera and J. R. Nayak, "Optimal solution of fuzzy nonlinear programming problems with linear constraints," *International Journal of Advances in Science and Technology*, vol. 4, no. 2, pp. 43-91, 2012.
- [6] C. MA, "A new exact penalty function method for nonlinear programming problems," *Science Asia*, vol. 44, p. 268, 2018, doi: 10.2306/scienceasia1513-1874.2018.44.268.
- [7] T. Antczak, "Optimality conditions for invex nonsmooth optimization problems with fuzzy objective functions," *Fuzzy Optimization and Decision Making*, vol. 22, pp. 1-21, 2023, doi: 10.1007/s10700-022-09381-4.
- [8] A. F. Jameel and A. Z. Radhi, "Penalty function method for solving fuzzy nonlinear programming problem," *International Journal of Recent Research in Mathematics Computer Science and Information Technology*, vol. 1, no. 1, pp. 1-13, 2014.
- [9] A. N. Gani and R. Yagarani, "Solving fuzzy linear programming problems by using the fuzzy exponential barrier method," *Ratio Mathematica*, vol. 46, 2023, doi: 10.23755/rm.v46i0.1088.
- [10] S. M. Othman and M. B. Abdulrazzaq, "Fuzzy logic system for drug storage based on the internet of things: a survey," *Indonesian Journal of Electrical Engineering and Computer Science*, vol. 29, no. 3, pp. 1382-1392, 2023, doi: 10.11591/ijeecs.v29.i3.pp1382-1392.
- [11] Z. A. Kanaya, "An interactive method for fuzzy multi objective nonlinear programming problems," *Journal of King Abdulaziz University-Science*, vol. 22, no. 1, pp. 103-112, 2010, doi: 10.4197/Sci.22-1.8.
- [12] P. Kaliyaperumal and A. Das, "A mathematical model for nonlinear optimization which attempts membership functions to address the uncertainties," *Mathematics*, vol. 10, no. 10, p. 1743, May 2022, doi: 10.3390/math10101743.
- [13] K. Kemal, "Interior and Exterior Penalty Methods to Solving Nonlinear Optimization Problem", *M.Sc. diss.*, Department of Mathematics, Addis Ababa University, Ethiopia, 2017.
- [14] M. Kiruthiga and T. Hemalatha, "A study on interactive fuzzy multi-objective nonlinear programming," *International Journal of Statistics and Applied Mathematics*, vol. 4, no. 4, pp. 59-65, 2019.
- [15] M. Lalitha and C. Loganathan, "A study on nonlinear programming problem in fuzzy environment," *International Organization of Scientific Research (IOSR) Journal of Mathematics*, vol. 10, no. 2, pp. 124-128, 2014, doi: 10.9790/5728-1022124128.
- [16] M. Lalitha and C. Loganathan, "Solving fully fuzzy nonlinear programming with inequality constraints," *International Journal of Mechanical Engineering and Technology (IJMET)*, vol. 8, no. 11, pp. 354-362, 2017.
- [17] A. Nagoorani and K. Sudha, "Optimality conditions for fuzzy non-linear unconstrained minimization problems," *Bulletin of Pure and Applied Sciences- Mathematics and Statistics*, vol. 38, no. 1, pp. 378-384, 2019, doi: 10.5958/2320-3226.2019.00041.9.
- [18] A. Allibhoy and J. Cortés, "Control Barrier Function Based Design of Gradient Flows for Constrained Nonlinear Programming," *IEEE Transactions on Automatic Control*, pp. 1-16, 2023, doi: 10.1109/TAC.2023.3306492.
- [19] U. M. Pirzada and V. D. Pathak, "Necessary and sufficient optimality conditions for nonlinear fuzzy optimization problem," *International Journal of Mathematical Science Education*, vol. 4, no. 1, pp. 1-16, 2011.
- [20] U. M. Pirzada and V. D. Pathak, "Newton method for solving the multi-variable fuzzy optimization problem," *Journal of Optimization Theory and Applications*, vol. 156, no. 3, pp. 867-881, 2013, doi: 10.1007/s10957-012-0141-3.
- [21] N. R. Shankar, G. A. Rao, J. M. Latha and V. Sireesha, "Solving a fuzzy nonlinear optimization problem by genetic algorithm," *International Journal Contemporary Mathematic Sciences*, vol. 5, no. 16, pp. 791-803, 2010.
- [22] A. K. Salman and A. S. Al-Jilawi, "On the bridge between exterior and interior penalty method," *Journal of Physics: Conference Series*, vol. 1818, no. 1, p. 012146, 2021, doi: 10.1088/1742-6596/1818/1/012146.
- [23] R. Saranya and P. Kaliyaperumal, "Fuzzy nonlinear programming problem for inequality constraints with alpha optimal solution in terms of trapezoidal membership functions," *International Journal of Pure and Applied Mathematics*, vol. 119, no. 9, pp. 53-63, 2018.
- [24] P. U. Maheswari and K. Ganesan, "A new approach for the solution of unconstrained fuzzy optimization problems," in *American Institute of Physics (AIP) Conference Proceeding*, vol. 2112, p. 020004, 2019, doi: 10.1063/1.5112189.
- [25] P. U. Maheswari and K. Ganesan, "A solution approach to fuzzy nonlinear programming problems," *International Journal of Pure and Applied Mathematics*, vol. 113, no. 13, pp. 291-300, 2017.
- [26] T. Lu and T. Liu, "Fuzzy nonlinear programming approach to the evaluation of manufacturing processes," *Engineering Applications of Artificial Intelligence*, vol. 72, pp. 183-189, Jun. 2018, doi: 10.1016/j.engappai.2018.04.003.
- [27] M. L. Puri and D. A. Ralescu, "Differential for fuzzy function," *Journal of Mathematical Analysis and Applications*, vol. 91, pp. 552-558, 1983, doi: 10.1016/0022-247X(83)90169-5.
- [28] S. K. Singh and S. P. Yadav, "Intuitionistic fuzzy non linear programming problems: modeling and optimization in manufacturing systems," *Journal of Intelligent and Fuzzy Systems*, vol. 28, no. 3, pp. 1421-1433, 2015, doi: 10.3233/IFS-141427.
- [29] M. Ma, M. Friedman, and A. Kandel, "A new fuzzy arithmetic," *Fuzzy Sets and Systems*, vol. 108, no. 1, pp. 83-90, 1999, doi: 10.1016/S0165-0114(97)00310-2.
- [30] D. Dubois and H. Prade, "Towards fuzzy differential calculus part 2: Integration on fuzzy intervals," *Fuzzy Sets and Systems*, vol. 8, no. 2, pp. 105-115, 1982, doi: 10.1016/0165-0114(82)90001-X.
- [31] R. Goetschel and W. Voxman, "Elementary fuzzy calculus," *Fuzzy Sets and Syst.*, vol. 18, no. 1, pp. 31-43, 1986, doi: 10.1016/0165-0114(86)90026-6.

BIOGRAPHIES OF AUTHORS

Vanaja Govindhasamy    is completed undergraduate and postgraduate degree in mathematics from Seethalakshmi Ramaswamy College, Trichy, Tamil Nadu. The B.Ed. degree in mathematics from Sri Ramakrishna Education College, Perambalur. Her research area is fuzzy optimization. Currently, she is working as a full-time researcher scholar at SRM Institute of Science and Technology, Kattankulathur, Tamil Nadu, India 603203. She can be contacted at email: vg5308@srmist.edu.in.



Ganesan Kandasamy    is currently works as a professor in the Department of Mathematics at SRM Institute of Science and Technology, Kattankulathur, Chennai, India. His area of research are fuzzy optimization, interval mathematics, operations research, ordinary, and partial differential equations. He has handled research projects funded by BRNS and has been visited country such Iran and Japan. There are about 19 candidates have obtained their Ph.D. degree under his guidance and many more are pursuing their research under his guidance at present so far. He is the author/co-author of over 135 research publications. He can be contacted at email: ganessank@srmist.edu.in.